

ON THE ARENS REGULARITY OF THE HERZ ALGEBRAS $A_p(G)$

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ABSTRACT. Let G be a locally compact group, $A_p(G)$ be the Herz algebra of G associated with $1 < p < \infty$. We show that $A_p(G)$ is Arens regular if and only if G is a discrete group and for each countable subgroup H of G , $A_p(H)$ is Arens regular. In the case G is a countable discrete group we investigate the relations between Arens regularity of $A_p(G)$ and the iterated limit condition. We consider the problem of Arens regularity of $l^1(G)$ as a subspace of $A_p(G)$. A few related results when the unit ball of $(l^1(G), \cdot, A_p(G))$ is bounded under $\|\cdot\|_1$ -norm are also determined.

1. INTRODUCTION

Let G be a locally compact group. For $1 < p < \infty$, let $A_p(G)$ denote the linear subspace of $C_0(G)$ consisting of all functions of the form

$$u = \sum_{i=1}^{\infty} g_i * f_i^{\vee}$$

where $f_i \in L_p(G)$, $g_i \in L_q(G)$, $\frac{1}{p} + \frac{1}{q} = 1$ and $\sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q < \infty$, $f^{\vee}(x) = f(x^{-1})$ for $x \in G$. $A_p(G)$ is a commutative Banach algebra with respect to the pointwise multiplication and the norm,

$$\|u\|_{A_p(G)} = \inf \left\{ \sum_{i=1}^{\infty} \|f_i\|_p \|g_i\|_q : u = \sum_{i=1}^{\infty} g_i * f_i^{\vee} \right\}.$$

When $p = 2$, $A_2(G) = A(G)$ is the Fourier algebra of G as introduced by Eymard [9]. The algebras $A_p(G)$ was introduced and studied by Herz [13]. Let $PM_p(G)$ denote the closure of $L_1(G)$, considered as an algebra of convolution operators on $L_p(G)$, with respect to the weak operator topology in the bounded operators on $L_p(G)$, denoted by $B(L_p(G))$. The space $PM_p(G)$ can be identified with the dual of $A_p(G)$ for each $1 < p < \infty$ [6].

Arens regularity of $A_p(G)$, has been studied by B. Forrest [10]. It is shown, for example, that if the algebra $A_p(G)$ is Arens regular then the group G is discrete. Furthermore, as shown in [15], Proposition 5.3, for $p = 2$ and G

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amenable, $A_p(G) = A(G)$ is Arens regular if and only if G is finite. In this paper, we shall study some problems related to Arens regularity of $A_p(G)$.

Notations and definitions which are necessary in the sequel are gathered in §2. In §3, it is shown that the Arens regularity of $A_p(G)$ can be reduced when G is discrete and finitely generated. Furthermore, we show that $A_p(G)$ is Arens regular if and only if $A_p(H)$ is Arens regular, for any countable subgroup H of G . In the case G is a discrete group, we consider the iterated limit condition for the Arens regularity of $A_p(G)$.

In [3] J. W. Baker and A. Rejali considered the Arens regularity of the algebras $l_1(S)$ where S is a discrete semigroup. Similarly, in Theorem 3.6 we consider the Arens regularity of $l_1(G)$ under various multiplications and norms. Furthermore, we show that $(l_1(G), \cdot, \|\cdot\|_{A_p(G)})$ is Arens regular if and only if $A_p(G)$ is Arens regular. In Theorem 3.7 we also show that if the unit ball of $(l_1(G), \cdot, \|\cdot\|_{A_p(G)})$ is bounded under $\|\cdot\|_1$, then G has no infinite Abelian subgroup.

2. PRELIMINARIES AND SOME NOTATIONS

Let G be a locally compact group with a fixed left Haar measure λ , which is also denoted by dx . The usual $L_p(G)$ ($1 \leq p \leq \infty$) are defined with respect to this measure λ . By $C(G)$ we denote the Banach space of bounded continuous complex valued functions endowed with the supremum norm. By $C_0(G)$ (resp. $C_{00}(G)$) we denote the subspace of $C(G)$ consisting of the functions on G vanishing at infinity (resp. with compact support).

For a Banach space X , we denote by X^* and X^{**} its first and second continuous duals. We regard X as naturally embedded into X^{**} . For $x \in X$ and $x^* \in X^*$, by $\langle x, x^* \rangle$ (or by $\langle x^*, x \rangle$) we denote the natural duality between X and X^* .

Let A be a Banach algebra. For $u \in A$ and $T \in A^*$ by uT (resp. Tu) we denote the element of A^* defined by $\langle uT, v \rangle = \langle T, vu \rangle$ (resp. $\langle Tu, v \rangle = \langle T, uv \rangle$), for v in A . When A is commutative it is clear that $uT = Tu$. Then A^{**} can be given two multiplications that extend the multiplication of A and for which A^{**} becomes a Banach algebra. When these two multiplications coincide on A^{**} , the algebra A is said to be Arens regular. Details of the constructions can be found in many places, including the book [4] and the papers [1, 2, 7, 17].

Let A be an arbitrary Banach algebra and A_1 be its closed unit ball. We call $T \in A^*$ weakly almost periodic if $O(T) = \{Tu : u \in A_1\}$ is relatively weakly compact in A^* . By $wap(A)$ we denote the linear subspace of A^* consisting of all weakly almost periodic functionals on A . The space $wap(A)$ is a closed subspace of A^* and by Theorem 1 of [7], A is Arens regular if and only if $wap(A) = A^*$ if and only if for each sequences $(a_n), (b_m)$ in A_1 and $T \in A^*$, the iterated limits

$$\lim_n \lim_m T(a_n b_m), \quad \lim_m \lim_n T(a_n b_m)$$

are equal, whenever both exist.

3. THE ARENS REGULARITY AND ITERATED LIMIT CONDITIONS

Forrest [10] showed that G is discrete, whenever $A_p(G)$ is Arens regular. In the following, we obtain a criteria for the Arens regularity of $A_p(G)$.

Theorem 3.1. *Let G be a locally compact group. Then $A_p(G)$ is Arens regular if and only if G is a discrete group and for each countable subgroup H of G , $A_p(H)$ is Arens regular.*

Proof. If $A_p(G)$ is Arens regular, then by [10, Theorem 3.2] G is discrete and for each subgroup H of G , $A_p(H)$ is Arens regular [10, Lemma 3.1].

Conversely, assume that G is a discrete and $A_p(H)$ is Arens regular for every countable subgroup H of G . Let $T \in PM_p(G)$. It is enough to show that $\{uT : u \in C_{00}(G), \|u\|_{A_p(G)} \leq 1\}$ is relatively weakly compact in $PM_p(G)$ [13, Proposition 3]. Now let u_n and v_m be two sequences in the unit ball of $A_p(G)$ with compact support, such that

$$a = \lim_n \lim_m T(u_n v_m), \quad b = \lim_m \lim_n T(u_n v_m)$$

both exist. Assume now that $S = (\cup_{i=1}^{\infty} \text{supp} u_i) \cup (\cup_{i=1}^{\infty} \text{supp} v_i)$. Let H be the subgroup of G generated by S . Then H is countable. Since $A_p(H)$ is Arens regular. So, $a = b$. It follows that $A_p(G)$ is Arens regular.

In [15] Lau and Wong have shown that, if $A_2(G)$ is Arens regular then G is finite, for a locally compact amenable group G . Furthermore, Forrest [11] showed that if G is a locally finite discrete group then, $A_p(G)$ is Arens regular if and only if G is finite. In the next theorem we show that the Arens regularity of $A_p(G)$ and finiteness of G can be considered only for finitely generated groups.

Theorem 3.2. *Let G be a locally compact group and $A_p(G)$ Arens regular. If the Arens regularity of $A_p(H)$ imply that H is finite, for every finitely generated subgroup H of G . Then G is finite.*

Proof. Suppose $A_p(G)$ is Arens regular, then G is discrete [10, Theorem 3.2]. Let H be a finitely generated subgroup of G . Then by [10] Lemma 3.1, $A_p(H)$ is Arens regular. Hence H is finite. So G is locally finite. It follows from [11] Proposition 3 that G is finite.

Let G be a countable discrete group. Then clearly $A_p(G)$ is separable. Hence the w^* -topology on the unit ball of $PM_p(G)$, the dual space of $A_p(G)$ is metrizable by [8], p. 426. So, it is sequentially compact, i.e each sequence in the unit ball of $PM_p(G)$ has a w^* -convergent subsequence. Let Y_p denote the unit ball of $PM_p(G)$ and

$$X_p = \text{span}\{\delta_x : x \in G\},$$

where $\delta_x(f) = f(x)$ for $f \in A_p(G)$, $x \in G$. Suppose $Z_p = X_p \cap Y_p$.

Lemma 3.3. *Let G be an amenable discrete group. Then*

$$w^* - clZ_p = Y_p.$$

Proof. Since G is amenable, so by [13, Theorem 5], $PM_p(G) = CONV_p(G)$ the set of all convolution operators on $L_p(G)$, and the conclusion holds.

Remark 3.4. If $(f_n), (g_m)$ are in the unit ball of $A_p(G)$ and (ϕ_k) in Y_p . Then clearly there are subsequences $(f'_n), (g'_m)$ and (ϕ'_k) of $(f_n), (g_m)$ and (ϕ_k) respectively, such that

$$\lim_n \lim_m \lim_k \phi'_k(f'_n g'_m), \quad \lim_m \lim_n \lim_k \phi'_k(f'_n g'_m)$$

both exist.

We now state the main result of this section.

Theorem 3.5. *Let G be an amenable countable discrete group. Then the following are equivalent:*

- (i) $A_p(G)$ is Arens regular.
- (ii) For each sequences $(f_n), (g_m)$ in the unit ball of $A_p(G)$, and (ϕ_k) in Z_p

$$\lim_n \lim_m \lim_k \phi_k(f_n g_m) = \lim_m \lim_n \lim_k \phi_k(f_n g_m)$$

whenever both exist.

- (iii) For each sequences $(f_n), (g_m)$ in the unit ball of $A_p(G)$ and $\phi_k \in Z_p$. Then

$$\{\phi_k(f_n g_m) : k > n > m\} \cap \{\phi_k(f_n g_m) : k > m > n\}^- \neq \emptyset.$$

Proof. (i) \implies (ii): Assume that (ϕ_k) is a sequence in Z_p . Then there exists a subsequence (ϕ_{k_s}) of (ϕ_k) and $\phi \in Y_p$ such that $\phi_{k_s} \rightarrow \phi$ in the w^* -topology of $PM_p(G)$. Thus we have,

$$\begin{aligned} a &= \lim_n \lim_m \lim_k \phi_k(f_n g_m) = \lim_n \lim_m \lim_s \phi_{k_s}(f_n g_m) = \lim_n \lim_m \phi(f_n g_m), \\ b &= \lim_m \lim_n \lim_k \phi_k(f_n g_m) = \lim_m \lim_n \lim_s \phi_{k_s}(f_n g_m) = \lim_m \lim_n \phi(f_n g_m). \end{aligned}$$

Since $A_p(G)$ is Arens regular, we have $a = b$.

(ii) \implies (i): Let $\phi \in Y_p$. Then by Lemma 3.3 there is a sequence (ϕ_k) in Z_p such that $\phi_k \rightarrow \phi$ in w^* -topology of $PM_p(G)$. Now let $(f_n), (g_m)$ be in the unit ball of $A_p(G)$ such that

$$a = \lim_n \lim_m \phi(f_n g_m), \quad b = \lim_m \lim_n \phi(f_n g_m).$$

Then we have

$$\begin{aligned} a &= \lim_n \lim_m \phi(f_n g_m) = \lim_n \lim_m \lim_k \phi_k(f_n g_m), \\ b &= \lim_m \lim_n \phi(f_n g_m) = \lim_m \lim_n \lim_k \phi_k(f_n g_m). \end{aligned}$$

So, $a = b$ and $A_p(G)$ is Arens regular.

(ii) \implies (iii): Let (ii) holds. Then (iii) follows immediately from Remark 3.4.

(iii) \implies (ii): It is clear.

Let G be a discrete group and $f \in l^1(G)$. Since we can write $f = \sum_{n=1}^{\infty} a_n \chi_{x_n}$. So $f = \sum_{n=1}^{\infty} a_n \chi_{x_n} * \chi_e^\vee$ and $f = \sum_{n=1}^{\infty} \|a_n \chi_{x_n}\|_q \|\chi_e\|_p = \sum_{n=1}^{\infty} |a_n| = \|f\|_1$. Whenever, χ_x denotes the characteristic function $1_{\{x\}}$. Therefore, $f \in A_p(G)$ and $\|f\|_{A_p} \leq \|f\|_1$. Furthermore, it is clear that for each $T \in l^\infty(G)$ and $f \in l^1(G)$, $Tf \in C_0(G)$. We now investigate the Arens regularity of $l^1(G)$ under various multiplications and norms.

Theorem 3.6. (i) $(l^1(G), \cdot, \|\cdot\|_1)$ is Arens regular.
(ii) $(l^1(G), \cdot, \|\cdot\|_1)$ is Arens regular if and only if G is finite.
(iii) $(l^1(G), \cdot, \|\cdot\|_{A_p})$ is not normed algebra in general.
(iv) $(l^1(G), \cdot, \|\cdot\|_{A_p(G)})$ is Arens regular if and only if $A_p(G)$ is Arens regular.

Proof. (i). Let (f_n) and (g_m) be two sequences in the unit ball of $l^1(G)$ and $T \in l^1(G)^*$ such that

$$\lim_n \lim_m T(f_n g_m), \quad \lim_m \lim_n T(f_n g_m)$$

both exist. Since $(l^1(G))_1$ is w^* -compact, so (f_n) and (g_m) have subnets w^* -converging to some f and g in the unit ball of $l^1(G)$, respectively. Since the original limits will be the same as the limits of the subnets. So we have,

$$\lim_n \lim_m T(f_n g_m) = \lim_n \lim_m g_m(Tf_n) = \lim_n g(Tf_n) = \lim_n f_n(Tg) = T(fg)$$

and similarly,

$$\lim_m \lim_n T(f_n g_m) = T(fg).$$

Hence by Theorem 1 of [7] $l^1(G)$ with pointwise multiplication is Arens regular.

(ii). See [18].

(iii). Let $G = \mathbb{Z}$. If $V_n = \{n, n+1, \dots, 0, 1, \dots, n\}$ and $\phi_n(k) = \frac{\chi_{V_n} * \chi_{V_n}^\vee(k)}{2n+1}$. Then $\phi_n \in A_p(\mathbb{Z}) \cap C_{00}(\mathbb{Z})$, $\|\phi_n\|_{A_p(\mathbb{Z})} = \phi_n(0) = 1$ and

$$\phi_n(k) = \begin{cases} \frac{2n+1-|k|}{2n+1} & |k| \leq 2n \\ 0 & |k| > 2n \end{cases}$$

But

$$\phi_2 * \phi_2^\vee(0) = \phi_2(0)^2 + 2 \sum_{k=1}^4 \phi_2(k)^2 = 1 + \frac{60}{25}.$$

So $\|\phi_2 * \phi_2^\vee\|_{A_p(\mathbb{Z})} > \|\phi_2\|_{A_p(\mathbb{Z})} \|\phi_2\|_{A_p(\mathbb{Z})}$. Therefore $(l^1(\mathbb{Z}), *, \|\cdot\|_{A_p(\mathbb{Z})})$ is not a normed algebra.

(iv) Since $l^1(G)$ is normed dense in $A_p(G)$. So, $l^1(G)$ is Arens regular if and only if $A_p(G)$ is Arens regular.

Put B_1 the unit ball of $(l^1(G), \cdot, \|\cdot\|_{A_p(G)})$.

Example 3.7. Let $G = \mathbb{Z}$ and ϕ_n be as above. Then $\phi \in B_1$ and

$$\phi = 1 + 2\left(\sum_{k=1}^{2n} \frac{2n+1-k}{2n+1}\right) = 1 + 2\frac{2n(2n+1)}{2(2n+1)} = 1 + 2n.$$

This shows that B_1 is unbounded with $\|\cdot\|_1$ -norm.

In the following we study the Arens regularity of $A_p(G)$, whenever B_1 is bounded under $\|\cdot\|_1$ -norm.

Theorem 3.8. *Let B_1 be bounded under $\|\cdot\|_1$ -norm. Then,*

- (i) $(l^1(G), \cdot, \|\cdot\|_{A_p(G)})$ is Arens regular.
- (ii) $A_p(G)$ is Arens regular.
- (iii) G has no infinite Abelian subgroup.
- (iv) There is no sequence $\{H_n\}$ of finite subgroups, such that

$$|H_1| < |H_2| < |H_3| < \dots$$

- (v) There is $M > 0$ such that for each $x \in G$, $O(x) \leq M$, where, $O(x)$ denote the order of element $x \in G$.

Proof. (i) Let $\phi \in (l^1(G), \cdot, \|\cdot\|_{A_p(G)})^*$ and $f \in l^1(G)$. Then

$$|\phi(f)| \leq \|\phi\| \|f\|_{A_p(G)} \leq \|\phi\| \|f\|_1,$$

so $\phi \in l^\infty(G)$. Now suppose $(f_n), (g_m) \in B_1$ and

$$\lim_n \lim_m \phi(f_n g_m), \quad \lim_m \lim_n \phi(f_n g_m)$$

both exist. Since B_1 is bounded under $\|\cdot\|_1$ -norm. Then there exists $\alpha > 0$ such that $\|f_n\| \leq \alpha$, $\|g_m\| \leq \alpha$ for all m, n . Since $(l^1(G), \cdot, \|\cdot\|_1)$ is Arens regular we have

$$\lim_n \lim_m \phi(f_n g_m) = \lim_m \lim_n \phi(f_n g_m).$$

Hence $(l^1(G), \cdot, \|\cdot\|_{A_p(G)})$ is Arens regular.

(ii) By 3.6(iv).

(iii) Let H be an infinite Abelian subgroup, and x_1, x_2, \dots , be a sequence of distinct element of H . Since $A_p(G)$ is Arens regular, so G is a periodic group. Hence each finitely generated subgroup H is finite. Now let H_n be a subgroup of H generated by $x_1, x_2, x_3, \dots, x_n$ and $\phi_n = \frac{\chi_{H_n} * \chi_{H_n}^\vee}{|H_n|}$. Then $\phi_n(e) = 1 = \|\phi_n\|_{A_p(G)}$ and

$$\phi_n(x) = \begin{cases} 1 & x \in H_n \\ 0 & x \notin H_n \end{cases}$$

So $\|\phi_n\|_1 = |H_n| \geq n$. This shows that B_1 is not bounded under $\|\cdot\|_1$ -norm.

(iv) If G has a sequence $\{H_n\}$ of finite subgroups, such that

$$|H_1| < |H_2| < |H_3| < \dots$$

Suppose $\phi_n = \frac{\chi_{H_n} * \chi_{H_n}^\vee}{|H_n|}$, then $\phi_n \in B_1$ and $\|\phi_n\|_1 \geq |H_n|$ which is a contradiction.

(v) This follows from (iv).

Corollary 3.9. *Let G be locally finite and B_1 be bounded under $\|\cdot\|_1$ -norm. Then G is finite.*

Proof. Let G be an infinite locally finite group. Then G has an infinite Abelian subgroup [14]. It follows from 3.7 (iii) that G must be finite.

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